

# ON COHOMOLOGY OF CRYSTALLOGRAPHIC GROUPS WITH CYCLIC HOLONOMY OF SPLIT TYPE

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**ABSTRACT.** We disprove a conjecture stating that the integral cohomology of any  $n$ -dimensional crystallographic group  $\mathbb{Z}^n \rtimes \mathbb{Z}_m$  admits a decomposition:

$$H^*(\mathbb{Z}^n \rtimes \mathbb{Z}_m) \cong \bigoplus_{i+j=*} H^i(\mathbb{Z}_m, H^j(\mathbb{Z}^n))$$

by providing a complete list of counterexamples up to dimension 5. We also find a counterexample with odd order holonomy,  $m = 9$ , in dimension 8 and finish the computations of the cohomology of 6-dimensional crystallographic groups arising as orbifold fundamental groups of certain Calabi-Yau toroidal orbifolds.

## 1. INTRODUCTION

An  $n$ -dimensional *crystallographic group*  $\Gamma$  is a discrete subgroup of isometries of  $\mathbb{R}^n$  acting properly discontinuously and cocompactly on  $\mathbb{R}^n$ . By the first Bieberbach theorem (see [4]), every such group has a normal subgroup  $L$  of translations which is a uniform lattice of  $\mathbb{R}^n$  and the holonomy group  $\Gamma/L$  is finite.

In [2], there is a complete structure theorem on the cohomology of crystallographic groups with cyclic holonomy of prime order. When such a group  $\Gamma$  contains a torsion element, i.e.  $\Gamma = \mathbb{Z}^n \rtimes \mathbb{Z}_p$ , the theorem asserts that the integral cohomology of  $\Gamma$  is given by the cohomology of  $\mathbb{Z}_p$  with coefficients in the cohomology of the lattice  $L$ . Also, it is conjectured that a similar decomposition holds for the cohomology of  $\Gamma = L \rtimes G$  for any finite cyclic group  $G$ .

**Conjecture 1.1** ([2, 5.2]). *Suppose that  $G$  is a finite cyclic group and  $L$  a finitely generated  $\mathbb{Z}G$ -lattice; then for any  $m \geq 0$  we have*

$$H^k(L \rtimes G, \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G, H^j(L, \mathbb{Z})).$$

We show that the conjecture already fails for a 4-dimensional crystallographic groups with holonomy  $\mathbb{Z}_4$  (see Corollary 3.3). This is the lowest possible dimension of a crystallographic group for which the conjecture is not true. In Section 4, we also compute the cohomology of all 4- and 5-dimensional crystallographic groups which do not satisfy the conjecture. There are 2 in dimension 4 both with holonomy  $\mathbb{Z}_4$  and in dimension 5, there are 5 with holonomy  $\mathbb{Z}_4$  and one with holonomy  $\mathbb{Z}_8$ . As further applications of our methods, in Section 5, we finish the computations of the cohomology of 6-dimensional crystallographic groups which arise as orbifold fundamental groups of certain toroidal orbifolds discussed in [2, Sec. 6]. Also, we give an example of an 8-dimensional crystallographic group with holonomy  $\mathbb{Z}_9$  which is the first counterexample with odd order holonomy.

Our approach is straightforward, as we compute both sides of the conjectured equation and immediately observe that they are not isomorphic. The method

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Both authors were supported by the Research Fund K.U.Leuven.

The first author was also supported by the FWO-Flanders Research Fellowship.

of computations is based on the so-called *twisted tensor product* construction introduced by Wall in [12]. Roughly stated, given an arbitrary group extension  $1 \rightarrow L \rightarrow \Gamma \rightarrow G \rightarrow 1$  and free resolutions  $B_*$  and  $C_*$  of  $\mathbb{Z}$  over  $\mathbb{Z}L$  and  $\mathbb{Z}G$  respectively, by inducting  $B_*$  to a resolution  $\text{Ind}_L^\Gamma B_*$  over  $\mathbb{Z}\Gamma$  and then tensoring with  $C_*$  over  $\mathbb{Z}G$ , assuming trivial right action on  $G$  on  $\text{Ind}_L^\Gamma B_*$ , one obtains an augmented chain complex of free  $\mathbb{Z}\Gamma$ -modules. Wall then proves that one can recursively construct new differentials of the complex to obtain an acyclic complex.

This method of computing the cohomology of crystallographic groups has already been implemented in GAP (see [7, Package HAP]). In Sections 2 and 3, we discuss how we adapt the algorithm to the case of crystallographic groups of split type. This shortens the computing times and allows us to find some counterexamples to the conjecture in dimensions 6 and 8.

Recently, using different methods, a 6-dimensional counterexample to the conjecture with holonomy  $\mathbb{Z}_4$  has been found by Langer and Lück ([8, 0.6]). In fact, they show that there is a counterexample to the conjecture for every holonomy group whose order is divisible by 4. They also verify the conjecture with an extra assumption that the action of  $G$  on  $L$  is free ([8, 0.5]).

In [10, 1.2], it was conjectured that the Lyndon-Hochschild-Serre spectral sequence associated to  $\mathbb{Z}^n \rtimes \mathbb{Z}_m$  collapses at  $E_2$  not only with integral coefficients, but more generally for all coefficient modules  $A$  that are  $\mathbb{Z}$ -free of finite rank having trivial  $\mathbb{Z}^n$ -action. We can easily show that several of the counterexamples to Conjecture 1.1 are also counterexamples to this conjecture. In Section 4 (see Theorem 4.2), we present a 3-dimensional counterexample to Conjecture [10, 1.2]. Interestingly, this is the lowest possible dimension for a crystallographic group with cyclic holonomy of split type whose associated Lyndon-Hochschild-Serre spectral sequence collapses with integral coefficients but does not collapse for some other coefficients  $A$ .

## 2. THE CRUXES OF THE METHOD

Before presenting the main steps used in our computations, first we discuss two notions that are essential to this method.

**2.1. Twisted tensor product.** Let  $1 \rightarrow L \rightarrow \Gamma \rightarrow G \rightarrow 1$  be an arbitrary extension of groups. Suppose  $(B_r, r \geq 0)$  and  $(C_s, s \geq 0)$  are free  $\mathbb{Z}L$  and  $\mathbb{Z}G$ -resolutions of  $\mathbb{Z}$ , respectively and denote by  $\partial_*$  the differential of  $C_*$ .

The induced module  $\text{Ind}_L^\Gamma B_* = \mathbb{Z}\Gamma \otimes_{\mathbb{Z}L} B_*$  is free over  $\mathbb{Z}\Gamma$ . Since induction is an exact functor,  $\text{Ind}_L^\Gamma B_*$  with the differentials induced from those of  $B_*$  becomes a free  $\mathbb{Z}\Gamma$ -resolution of  $\mathbb{Z}G$ .

Next, let us endow each module  $\text{Ind}_L^\Gamma B_*$  with the trivial right  $G$ -action and define:

$$A_{r,s} := \text{Ind}_L^\Gamma B_r \otimes_{\mathbb{Z}G} C_s.$$

Set  $\alpha_s = \text{rk}_{\mathbb{Z}G}(C_s)$  and denote by  $\text{Ind}_L^\Gamma B$  the graded complex  $\bigoplus_r \text{Ind}_L^\Gamma B_r$  and let  $\epsilon$  be its augmentation. Then

$$D_s := \bigoplus_r A_{r,s} = \text{Ind}_L^\Gamma B \otimes_{\mathbb{Z}G} C_s$$

is a direct sum of  $\alpha_s$  copies of  $\text{Ind}_L^\Gamma B$ , which together with augmentation  $\epsilon_s := (\text{Id} \otimes_{\mathbb{Z}L} \epsilon)^{\alpha_s}$  onto  $C_s$  entails a free  $\mathbb{Z}\Gamma$ -resolution of  $C_s$ . Lastly, we denote by  $d_0$  the differential of each complex  $D_s$  and define:

$$A := \bigoplus_s D_s = \bigoplus_{r,s} A_{r,s}$$

graded by  $r + s$ .

The following crucial result was proven in [12]. In fact, its proof will comprise the main steps of the algorithm which we will discuss later.

**Theorem 2.1** ([12, Lem. 2, Th. 1]). *There exist  $\mathbb{Z}\Gamma$ -homomorphism  $d_k: A_{r,s} \rightarrow A_{r+k-1,s-k}$  ( $k \geq 1, s \geq k$ ) such that*

- (i)  $\epsilon_{s-1}d_1 = \partial\epsilon_s: A_{0,s} \rightarrow C_{s-1}$
- (ii)  $\sum_{i=0}^k d_i d_{k-i} = 0$ , for each  $k$ , (where  $d_{k|A_{r,s}}$  is interpreted as zero if  $r = k = 0$  or if  $s < k$ .)

Moreover, with the differential  $d = \sum_{k=0}^{\infty} d_k$ , the complex  $(A, d)$  is acyclic and hence it yields a free  $\mathbb{Z}\Gamma$ -resolution of  $\mathbb{Z}$ .

**2.2. Contracting homotopies.** Let  $(Q, d)$  be an acyclic chain complex. It will be often necessary to take preimages of  $d$  for elements which are in  $\ker d = \operatorname{im} d$ . A suitable computational method for this is by using a contracting homotopy. More about this approach could be found in [6, Section 3].

A *contracting homotopy* of an acyclic complex  $Q$  is a chain map  $h: Q_i \rightarrow Q_{i+1}$  such that  $hd + dh = \operatorname{Id}$ . Then for each  $y \in \ker d$ , we have  $dh(y) = y$ . So  $h$  maps such an element  $y$  to its preimage under  $d$ .

Contracting homotopies are often easy to construct. To obtain a contracting homotopy for a  $\mathbb{Z}[\mathbb{Z}^n]$ -resolution  $B$  of  $\mathbb{Z}$ , we will use the standard formula given in [3, p. 214], which provides a contracting homotopy for a tensor product of acyclic complexes equipped with contracting homotopies.

Let  $L = \mathbb{Z}^n$  and  $\Gamma = L \rtimes G$ . We need to explain how to define a contracting homotopy on the induced complex  $\operatorname{Ind}_L^\Gamma B$  from a given contracting homotopy  $h$  on  $B$ . Since every element of  $\operatorname{Ind}_L^\Gamma B$  can be written as a direct sum of elements of the form  $(1, g) \otimes_{\mathbb{Z}L} y$  for  $(1, g) \in \Gamma$  and  $y \in B$ , we define the contracting homotopy by:

$$f: \operatorname{Ind}_L^\Gamma B \rightarrow \operatorname{Ind}_L^\Gamma B, \quad (1, g) \otimes_{\mathbb{Z}L} y \mapsto (1, g) \otimes_{\mathbb{Z}L} h(y).$$

**2.3. The steps involved.** We are now ready to describe the key steps of the algorithm used to compute the cohomology of an  $n$ -dimensional crystallographic group  $\Gamma = L \rtimes G$ . The reader may find it helpful to refer to the next section where we explicitly implement these steps in a specific example.

To obtain the free  $\mathbb{Z}L$ -resolution  $B_*$  of  $\mathbb{Z}$ , we tessellate  $\mathbb{R}^n$  into standard  $n$ -cubes of length 1. This defines an  $L$ -equivariant CW-structure on  $\mathbb{R}^n$  and the associated chain complex yields the desired resolution.

We denote by  $t_i$  for  $1 \leq i \leq n$  the generators of  $L = \mathbb{Z}^n$  which correspond to translations by 1 in the coordinate  $i$ . We denote by  $e$  the origin of  $\mathbb{R}^n$ , by  $e_i$  for  $1 \leq i \leq n$  the 1-dimensional segment from  $e$  to  $t_i e$ , and by  $e_{i_1 i_2 \dots i_m}$  the  $m$ -dimensional cube spanned by  $e_{i_1}, e_{i_2}, \dots, e_{i_m}$ . Then

$$B_m = \langle e_{i_1 \dots i_m}, 1 \leq i_1 < \dots < i_m \leq n \rangle_{\mathbb{Z}L} \quad \text{for } 0 \leq m \leq n$$

and the differentials of  $B_*$ , denoted by  $d_*^B$ , are given by:

$$d_m^B(e_{i_1 \dots i_m}) = \sum_{j=1}^m (-1)^{j-1} (t_{i_j} - 1) e_{i_1 \dots \hat{i}_j \dots i_m}.$$

Next, we need a free  $\mathbb{Z}G$ -resolution  $C_*$  of  $\mathbb{Z}$ .

*Remark 1.* In our computations, the holonomy will always be a finite cyclic group, i.e.  $G = \langle x \mid x^q = 1 \rangle$ . In this case, we will take for  $C_*$  the standard 2-periodic resolution  $C_i = \mathbb{Z}G$  for all  $i \geq 0$  and  $\partial_{i+1}: C_{i+1} \xrightarrow{x-1} C_i$  when  $i$  is even and  $\partial_{i+1}: C_{i+1} \xrightarrow{x^{q-1} + \dots + x + 1} C_i$  when  $i$  is odd.

Now, using twisted tensor product construction, we obtain a free  $\mathbb{Z}\Gamma$ -resolution  $(A, d)$  of  $\mathbb{Z}$  as follows:

- (i) As discussed, we construct the resolutions  $(C_*, \partial)$  and  $(\text{Ind}_L^\Gamma B_*, \text{Id} \otimes_{\mathbb{Z}L} d_*^B)$ , and free  $\mathbb{Z}\Gamma$ -modules  $A_{r,s}$  for each  $0 \leq r \leq n$  and  $0 \leq s$  and set  $A_m = \bigoplus_{r+s=m} A_{r,s}$ .
- (ii) For  $n = 1$ , we define a contracting homotopy  ${}^1h$  on the  $\mathbb{Z}[\mathbb{Z}]$ -free resolution  $(B_*, d_*^B, n = 1)$  by  ${}^1h(1) = e$  and

$${}^1h(t_1^j e) = \begin{cases} \sum_{i=0}^{j-1} t_1^i e_{i_1 \dots i_m} & j > 0 \\ -\sum_{i=1}^{-j} t_1^{-i} e_{i_1 \dots i_m} & j < 0 \\ 0 & j = 0. \end{cases}$$

For each  $k \geq 1$ , since  $(B_*, d_*^B, n = k+1)$  is isomorphic to the tensor product of  $(B_*, d_*^B, n = k)$  and the above resolution, we can and will define a contracting homotopy  $h : B \rightarrow B$  by the recursive formula (see [3, p. 214]):

$${}^{k+1}h = {}^k h \otimes \iota + ({}^k h \epsilon) \otimes {}^1h,$$

where  $\iota$  is the identity map on  $(B_*, d_*^B, n = 1)$  and  ${}^k h$  is the contracting homotopy on  $(B_*, d_*^B, n = k)$ .

- (iii) Let  $r = 0$  and  $\beta$  be a generator of  $A_{0,s}$ . We define  $d_1(\beta) = f(\partial(\epsilon_s(\beta))) \subseteq A_{0,s-1}$ .

For  $r = 1$ , we have that  $\epsilon_{s-1} d_1 d_0 = \partial \epsilon_s d_0 = 0$ . Hence,  $d_1 d_0 : A_{1,s} \rightarrow A_{0,s-1}$  maps into  $\ker \epsilon_s = \text{im } d_0$ .

- (iv) So, for any generator  $\beta \in A_{1,s}$ , we define  $d_1(\beta) = -f(d_1(d_0(\beta)))$ . Similar occurs for  $r \geq 2$  and for any generator  $\beta \in A_{2,s}$ , we define  $d_1(\beta) = -f(d_1(d_0(\beta)))$ .

For  $k \geq 2$  we need to define  $d_k$  which satisfy the equation  $\sum_{i=0}^k d_i d_{k-i} = 0$ . Suppose, we defined  $d_i$  for  $i < k$  and  $d_k|_{A_{r-1,s}}$  satisfying this property. It is not difficult to check that  $\sum_{i=1}^k d_i d_{k-i}$  is in  $\ker d_0 = \text{im } d_0$  (see Lemma 2 of [12]).

- (v) Then, for a generator  $\beta \in A_{r,s}$  we take  $d_k(\beta) = -f(\sum_{i=1}^k d_i d_{k-i})(\beta)$ .

This yields the free  $\mathbb{Z}\Gamma$ -resolution  $(A, d)$ . To calculate the cohomology of  $\Gamma$  we:

- (vi) apply the functor  $\text{Hom}_{\mathbb{Z}\Gamma}(-, \mathbb{Z})$  to  $(A, d)$  to obtain a cochain complex of finitely generated  $\mathbb{Z}$ -free modules  $(F, \delta)$ .
- (vii) For each  $0 \leq i \leq n+1$ , reduce the matrix representing the boundary map  $\delta_i : F_i \rightarrow F_{i+1}$  to Smith normal form and read off the cohomology group  $H^{i+1}(\Gamma)$  via the isomorphism:

$$F_{i+1}/\text{Im } \delta_i \cong H^{i+1}(\Gamma) \oplus \text{Im } \delta_{i+1}.$$

*Remark 2.* When  $G$  is a cyclic group and  $C_*$  is its standard 2-periodic resolution, since the resolution  $(B, d^B)$  has length  $n$ , one can easily observe that the resolution  $(A, d)$  will also be 2-periodic starting from dimension  $n+1$ . So, in all the steps we can stop the computations once we reach this dimension.

### 3. A COUNTEREXAMPLE

In this section, we provide a counterexample to Conjecture 1.1, by applying the computational steps of Section 2.

Let  $\Gamma$  be a 4-dimensional crystallographic group  $L \rtimes G$  where  $G = \langle M \mid M^4 = 1 \rangle$  is the cyclic of order 4 acting on  $L = \mathbb{Z}^4$  by a left multiplication given by the matrix:

$$(3.1) \quad M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Proposition 3.1.** *The integral cohomology of  $\Gamma$  is as follows:*

$$H^i(\Gamma) = \begin{cases} \mathbb{Z} & i = 1 \\ \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 & i = 2 \\ \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 & i = 3 \\ \mathbb{Z}_4^2 & i = 2k, \quad k \geq 2 \\ \mathbb{Z}_2^4 & i = 2k + 1, \quad k \geq 2 \end{cases}$$

*Proof.* Let  $(B, d^B)$  be the free  $\mathbb{Z}L$ -resolution of  $\mathbb{Z}$  defined in 2.3 for  $n = 4$ .

To simplify the notation we introduce the symbol:

$$\mathfrak{C}(j, t_k, e_{i_1 \dots i_m}) := \begin{cases} \sum_{i=0}^{j-1} t_k^i e_{i_1 \dots i_m} & j > 0 \\ -\sum_{i=1}^j t_k^{-i} e_{i_1 \dots i_m} & j < 0 \\ 0 & j = 0 \end{cases}$$

Then, the contracting homotopy  $h: B \rightarrow B$  of the augmented resolution  $B$  (where we set  $B_{-1} = \mathbb{Z}$ ) is given by:

$$\begin{aligned} h(1) &= e, \\ h(t_1^i t_2^j t_3^k t_4^l e) &= t_2^j t_3^k t_4^l \mathfrak{C}(i, t_1, e_1) + t_3^k t_4^l \mathfrak{C}(j, t_2, e_2) \\ &\quad + t_4^l \mathfrak{C}(k, t_3, e_3) + \mathfrak{C}(l, t_4, e_4), \\ h(t_1^i t_2^j t_3^k t_4^l e_1) &= 0, \\ h(t_1^i t_2^j t_3^k t_4^l e_2) &= t_2^j t_3^k t_4^l \mathfrak{C}(i, t_1, e_1 e_2), \\ h(t_1^i t_2^j t_3^k t_4^l e_3) &= t_2^j t_3^k t_4^l \mathfrak{C}(i, t_1, e_1 e_3) + t_3^k t_4^l \mathfrak{C}(j, t_2, e_2 e_3), \\ h(t_1^i t_2^j t_3^k t_4^l e_4) &= t_2^j t_3^k t_4^l \mathfrak{C}(i, t_1, e_1 e_4) + t_3^k t_4^l \mathfrak{C}(j, t_2, e_2 e_4) \\ &\quad + t_4^l \mathfrak{C}(k, t_3, e_3 e_4), \\ h(t_1^i t_2^j t_3^k t_4^l e_1 e_2) &= h(t_1^i t_2^j t_3^k t_4^l e_1 e_3) = h(t_1^i t_2^j t_3^k t_4^l e_1 e_4) = 0 \\ h(t_1^i t_2^j t_3^k t_4^l e_2 e_3) &= t_2^j t_3^k t_4^l \mathfrak{C}(i, t_1, e_1 e_2 e_3), \\ h(t_1^i t_2^j t_3^k t_4^l e_2 e_4) &= t_2^j t_3^k t_4^l \mathfrak{C}(i, t_1, e_1 e_2 e_4), \\ h(t_1^i t_2^j t_3^k t_4^l e_3 e_4) &= t_2^j t_3^k t_4^l \mathfrak{C}(i, t_1, e_1 e_3 e_4) \\ &\quad + t_3^k t_4^l \mathfrak{C}(j, t_2, e_2 e_3 e_4), \\ h(t_1^i t_2^j t_3^k t_4^l e_1 e_2 e_3) &= h(t_1^i t_2^j t_3^k t_4^l e_1 e_2 e_4) = h(t_1^i t_2^j t_3^k t_4^l e_1 e_3 e_4) = 0, \\ h(t_1^i t_2^j t_3^k t_4^l e_2 e_3 e_4) &= t_2^j t_3^k t_4^l \mathfrak{C}(i, t_1, e_1 e_2 e_3 e_4). \end{aligned}$$

Next, we set  $x = (1, M) \in L \rtimes G$  and consider the standard  $\mathbb{Z}G$ -resolution  $(C, \partial)$  defined in Remark 1:

$$(3.2) \quad \cdots \rightarrow \mathbb{Z}[x]/(x^4 - 1) \xrightarrow{x^3 + x^2 + x + 1} \mathbb{Z}[x]/(x^4 - 1) \xrightarrow{x - 1} \mathbb{Z}[x]/(x^4 - 1) \rightarrow \mathbb{Z} \rightarrow 0.$$

Since, every  $C_i$  is 1-generated as a  $\mathbb{Z}G$ -module, by construction:

$$D_s = \bigoplus_r A_{r,s} \cong \text{Ind}_L^\Gamma B$$

for each  $s \geq 0$ .

Occasionally, we will add a superscript to generators of  $B$  to denote to which  $A_{r,s}$  they belong, i.e.

$$1 \otimes_L e_{i_1 i_2 \dots i_r}^s \in A_{r,s} \quad \text{for } r, s \geq 0, 1 \leq i_1 < \dots < i_r \leq 4$$

and from now on, we will simplify the notation by setting:

$$ge_{i_1 \dots i_r}^s := g \otimes_L e_{i_1 \dots i_r}^s.$$

Now, using the recursive steps (iii)-(v) discussed in Section 2.3, we compute the maps  $d_k: A_{r,s} \rightarrow A_{r+k-1,s-k}$  of Theorem 2.1.

Let  $s \geq 1$ . Then,

$$\begin{aligned} d_1(e^{2s-1}) &= (x-1)e \\ d_1(e^{2s}) &= (x^3 + x^2 + x + 1)e \end{aligned}$$

$$\begin{aligned} d_1(e_1^{2s-1}) &= e_1 - xe_2 \\ d_1(e_2^{2s-1}) &= xt_1^{-1}t_4e_1 + e_2 - xe_4 \\ d_1(e_3^{2s-1}) &= xt_3^{-1}t_4e_3 + e_3 - xe_4 \\ d_1(e_4^{2s-1}) &= -xe_4 + e_4 \end{aligned}$$

$$\begin{aligned} d_1(e_1^{2s}) &= x^2t_1^{-1}t_4e_1 - e_1 + x^3t_2^{-1}t_4e_2 - xe_2 - x^3e_4 - x^2e_4 \\ d_1(e_2^{2s}) &= -x^3e_1 + xt_1^{-1}t_4e_1 + x^2t_2^{-1}t_4e_2 - e_2 - x^2e_4 - xe_4 \\ d_1(e_3^{2s}) &= x^3t_3^{-1}t_4e_3 - x^2e_3 + xt_3^{-1}t_4e_3 - e_3 - x^3e_4 - xe_4 \\ d_1(e_4^{2s}) &= -(x^3 + x^2 + x + 1)e_4 \end{aligned}$$

$$\begin{aligned} d_1(e_{12}^{2s-1}) &= xt_1^{-1}t_4e_{12} - e_{12} + xe_{24} \\ d_1(e_{13}^{2s-1}) &= -e_{13} - xt_3^{-1}t_4e_{23} + xe_{24} \\ d_1(e_{14}^{2s-1}) &= -e_{14} + xe_{24} \\ d_1(e_{23}^{2s-1}) &= xt_1^{-1}t_3^{-1}t_4^2e_{13} - xt_1^{-1}t_4e_{14} - e_{23} + xt_3^{-1}t_4e_{34} \\ d_1(e_{24}^{2s-1}) &= -xt_1^{-1}t_4e_{14} - e_{24} \\ d_1(e_{34}^{2s-1}) &= -xt_3^{-1}t_4e_{34} - e_{34} \end{aligned}$$

$$\begin{aligned} d_1(e_{12}^{2s}) &= x^3t_2^{-1}t_4e_{12} + x^2t_1^{-1}t_2^{-1}t_4^2e_{12} + xt_1^{-1}t_4e_{12} + e_{12} \\ &\quad - x^3e_{14} - x^2t_1^{-1}t_4e_{14} + x^2t_2^{-1}t_4e_{24} + xe_{24} \\ d_1(e_{13}^{2s}) &= -x^2t_1^{-1}t_4e_{13} + e_{13} + x^3t_2^{-1}t_3^{-1}t_4^2e_{23} - xt_3^{-1}t_4e_{23} \\ &\quad - x^3t_2^{-1}t_4e_{24} + xe_{24} + x^3t_3^{-1}t_4e_{34} - x^2e_{34} \\ d_1(e_{14}^{2s}) &= -x^2t_1^{-1}t_4e_{14} + e_{14} - x^3t_2^{-1}t_4e_{24} + xe_{24} \\ d_1(e_{23}^{2s}) &= -x^3t_3^{-1}t_4e_{13} + xt_1^{-1}t_3^{-1}t_4^2e_{13} + x^3e_{14} - xt_1^{-1}t_4e_{14} \\ &\quad - x^2t_2^{-1}t_4e_{23} + e_{23} - x^2e_{34} + xt_3^{-1}t_4e_{34} \\ d_1(e_{24}^{2s}) &= -x^2t_2^{-1}t_4e_{24} + e_{24} + x^3e_{14} - xt_1^{-1}t_4e_{14} \\ d_1(e_{34}^{2s}) &= -x^3t_3^{-1}t_4e_{34} + x^2e_{34} - xt_3^{-1}t_4e_{34} + e_{34} \end{aligned}$$

$$\begin{aligned}
d_1(e_{123}^{2s-1}) &= xt_1^{-1}t_3^{-1}t_4^2e_{123} + e_{123}^{2s-2} - xt_1^{-1}t_4e_{124} - xt_3^{-1}t_4e_{234} \\
d_1(e_{124}^{2s-1}) &= -xt_1^{-1}t_4e_{124} + e_{124} \\
d_1(e_{134}^{2s-1}) &= e_{134} + xt_3^{-1}t_4e_{234} \\
d_1(e_{234}^{2s-1}) &= -xt_1^{-1}t_3^{-1}t_4^2e_{134} + e_{234} \\
d_1(e_{123}^{2s}) &= x^3t_2^{-1}t_3^{-1}t_4^2e_{123} - x^2t_1^{-1}t_2^{-1}t_4^2e_{123} + xt_1^{-1}t_3^{-1}t_4^2e_{123} - e_{123} \\
&\quad - x^3t_2^{-1}t_4e_{124} - xt_1^{-1}t_4e_{124} + x^3t_3^{-1}t_4e_{134} - x^2t_1^{-1}t_4e_{134} \\
&\quad + x^2t_2^{-1}t_4e_{234} - xt_3^{-1}t_4e_{234} \\
d_1(e_{124}^{2s}) &= -x^3t_2^{-1}t_4e_{124} - x^2t_1^{-1}t_2^{-1}t_4^2e_{124} - xt_1^{-1}t_4e_{124} - e_{124} \\
d_1(e_{134}^{2s}) &= x^2t_1^{-1}t_4e_{134} - e_{134}^{2s-1} - x^3t_2^{-1}t_3^{-1}t_4^2e_{234} + xt_3^{-1}t_4e_{234} \\
d_1(e_{234}^{2s}) &= x^3t_3^{-1}t_4e_{134} - xt_1^{-1}t_3^{-1}t_4^2e_{134} + x^2t_2^{-1}t_4e_{234} - e_{234} \\
d_1(e_{1234}^{2s-1}) &= (-xt_1^{-1}t_3^{-1}t_4^2 - 1)e_{1234} \\
d_1(e_{1234}^{2s}) &= (-x^3t_2^{-1}t_3^{-1}t_4^2 + x^2t_1^{-1}t_2^{-1}t_4^2 - xt_1^{-1}t_3^{-1}t_4^2 + 1)e_{1234}
\end{aligned}$$

For  $d_2$  we obtain:

$$d_2(e^s) = 0$$

$$\begin{aligned}
d_2(e_1^{2s}) &= e_{14} \\
d_2(e_2^{2s}) &= x^3e_{14} \\
d_2(e_3^{2s}) &= x^2e_{34} + e_{34} \\
d_2(e_4^{2s}) &= 0
\end{aligned}$$

$$\begin{aligned}
d_2(e_i^{2s+1}) &= 0 \quad \text{for } i = 1, 4 \\
d_2(e_2^{2s+1}) &= x^3e_{14} + e_{24} \\
d_2(e_3^{2s+1}) &= x^2e_{34} + e_{34}
\end{aligned}$$

$$\begin{aligned}
d_2(e_{i4}^{2s}) &= 0 \quad \text{for } i = 1, 2, 3 \\
d_2(e_{12}^{2s}) &= -x^3t_2^{-1}t_4e_{124} - e_{124} \\
d_2(e_{13}^{2s}) &= x^2t_1^{-1}t_4e_{134} - t_4e_{134} - e_{134} \\
d_2(e_{23}^{2s}) &= x^3t_3^{-1}t_4e_{134} + x^2t_2^{-1}t_4e_{234} - e_{234}
\end{aligned}$$

$$\begin{aligned}
d_2(e_{i4}^{2s+1}) &= 0 \quad \text{for } i = 1, 2, 3 \\
d_2(e_{12}^{2s+1}) &= -x^3t_2^{-1}t_4e_{124} - e_{124} \\
d_2(e_{13}^{2s+1}) &= x^2t_1^{-1}t_4e_{134} - e_{134} \\
d_2(e_{23}^{2s+1}) &= x^3t_3^{-1}t_4e_{134} + x^2t_2^{-1}t_4e_{234} - t_4e_{234} - e_{234} \\
d_2(e_{123}^{s+1}) &= (-x^3t_2^{-1}t_3^{-1}t_4^2 + x^2t_1^{-1}t_2^{-1}t_4^2 + t_4 + 1)e_{1234} \\
d_2(e_{i_1i_2i_3}^s) &= 0 \quad \text{for } (i_1, i_2, i_3) \neq (1, 2, 3)
\end{aligned}$$

$$d_2(e_{1234}^s) = 0$$

$$d_k \equiv 0 \quad \text{for } k \geq 3.$$

Applying the functor  $\text{Hom}_{\mathbb{Z}\Gamma}(-, \mathbb{Z})$  to the resolution  $(A, d)$ , we obtain a complex  $(F, \delta)$  with dimensions:

$$\dim(F_i) = \begin{cases} 1 & i = 0 \\ 5 & i = 1 \\ 11 & i = 2 \\ 15 & i = 3 \\ 16 & i \geq 4. \end{cases}$$

After numbering the generators of  $F$  in lexicographical order, we determine the matrices representing the differentials and reduce them to Smith normal form (SNF):

	Diagonal of SNF
$\delta_1$	$[0]$
$\delta_2$	$[1, 1, 2, 4, 0]$
$\delta_3$	$[1, 1, 1, 1, 2, 4, 0, 0, 0, 0, 0]$
$\delta_4$	$[1, 1, 1, 1, 1, 1, 4, 4, 0, 0, 0, 0, 0, 0, 0]$
$\delta_{2k-1}, k \geq 3$	$[1, 1, 1, 1, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0]$
$\delta_{2k}, k \geq 3$	$[1, 1, 1, 1, 1, 1, 4, 4, 0, 0, 0, 0, 0, 0, 0]$

Using step (vii), we finish the computations of the cohomology of  $\Gamma$ . □

Next, we compute the right hand side of the conjectured equation in 1.1 for the group  $\Gamma$ .

**Proposition 3.2.** *The following holds.*

$$H^i(G, H^j(L, \mathbb{Z})) = \begin{cases} \mathbb{Z} & 0 \leq j \leq 3, i = 0 \\ \mathbb{Z}_2 & j = 1, 3, i \geq 1 \\ \mathbb{Z}_2 & j = 2, i \geq 1, 2 \nmid i \\ \mathbb{Z}_2 & j = 4, 2 \nmid i \\ \mathbb{Z}_4 & j = 0, i \geq 1, 2 \mid i \\ \mathbb{Z}_4 & j = 2, 2 \nmid i \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Note that  $H^1(L, \mathbb{Z}) \cong \text{Hom}(L, \mathbb{Z}) \cong \mathbb{Z}^4$ . Let it be generated by  $t_i, 1 \leq i \leq 4$ . We interpret  $H^j(L, \mathbb{Z})$  as  $j$ -th exterior power  $\Lambda^j(H^1(L, \mathbb{Z}))$  with generators  $t_{i_1 \dots i_j} := t_{i_1} \wedge \dots \wedge t_{i_j}$  for  $1 \leq i_1 < \dots < i_j \leq 4$ .

The action of  $G$  on  $H^j(L, \mathbb{Z})$  is given by:

$$t_{i_1 \dots i_j} \cdot M = t_{i_1} M^T \wedge t_{i_2} M^T \wedge \dots \wedge t_{i_j} M^T$$



Arranging generators of  $H^j(L, \mathbb{Z})$  in lexicographical order, we obtain the following matrices for the action of  $M$  on  $H^j(L, \mathbb{Z})$ :

$$\begin{aligned}
 j = 1 : & \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\
 j = 2 : & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \\
 j = 3 : & \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \\
 j = 4 : & [-1]
 \end{aligned}$$

Applying  $\text{Hom}_{\mathbb{Z}G}(-, H^j(L, \mathbb{Z}))$  to the resolution (3.2) for  $G$ , we obtain a complex:

$$0 \rightarrow H^j(L, \mathbb{Z}) \xrightarrow{M-I} H^j(L, \mathbb{Z}) \xrightarrow{M^3+M^2+M+I} H^j(L, \mathbb{Z}) \xrightarrow{M-I} \dots$$

with the corresponding Smith normal forms:

$$\begin{aligned}
 j = 1 : \text{SNF}(M - I) &= \text{diag}([1, 1, 2, 0])_{4 \times 4}, \\
 \text{SNF}(M^3 + M^2 + M + 1) &= \text{diag}([2, 0, 0, 0])_{4 \times 4} \\
 j = 2 : \text{SNF}(M - I) &= \text{diag}([1, 1, 1, 1, 4, 0])_{6 \times 6}, \\
 \text{SNF}(M^3 + M^2 + M + 1) &= \text{diag}([2, 0, 0, 0, 0, 0])_{6 \times 6} \\
 j = 3 : \text{SNF}(M - I) &= \text{diag}([1, 1, 2, 0])_{4 \times 4}, \\
 \text{SNF}(M^3 + M^2 + M + 1) &= \text{diag}([2, 0, 0, 0])_{4 \times 4} \\
 j = 4 : \text{SNF}(M - I) &= [2], \\
 \text{SNF}(M^3 + M^2 + M + 1) &= [0]
 \end{aligned}$$

Proceeding as in step (vii), we finish the computations.  $\square$

From the two propositions, we immediately obtain:

**Corollary 3.3.** *For the crystallographic group  $\Gamma$ , we have:*

$$H^4(\Gamma, \mathbb{Z}) = \mathbb{Z}_4^2 \neq \mathbb{Z}_4 \oplus \mathbb{Z}_2^3 = \bigoplus_{i+j=4} H^i(G, H^j(L, \mathbb{Z}))$$

Therefore, Conjecture 1.1 is false.

#### 4. ALL COUNTEREXAMPLES UP TO DIMENSION 5

The algorithm for twisted tensor product is implemented, for example, in the HAP package in the system GAP (see [7]). We implemented our version of the algorithm which is adjusted to our case and allows for more efficient computations.

In this section we list all cases of crystallographic groups of dimensions up to 5 which do not satisfy Conjecture 1.1. For the list of all crystallographic groups in low dimensions we use the classification given in CARAT (see [9]).

Up to dimension 3, all crystallographic groups of the form  $L \rtimes G$  with  $G$  being cyclic satisfy the conjecture.

In dimension 4, there are 44 non-isomorphic crystallographic groups of this type. Among these, 2 do not satisfy Conjecture 1.1. Both of them have the holonomy group of order 4.

*Remark 3.* The holonomy representation of the first group is generated by the matrix:

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

which has the cohomology given in the Table 1 under the notation  $1 : 4 \rtimes 4$ . We observe that:

$$\begin{aligned} H^4(\mathbb{Z}^4 \rtimes \mathbb{Z}_4, \mathbb{Z}) &= \mathbb{Z}_4 \oplus \mathbb{Z}_2^3, \\ \bigoplus_{i+j=4} H^i(G, H^j(\mathbb{Z}^4, \mathbb{Z})) &= \mathbb{Z}_4^2 \oplus \mathbb{Z}_2^2. \end{aligned}$$

This implies that, in the associated Lyndon-Hochschild-Serre spectral sequence, there are nonzero differentials.

*Remark 4.* Let us note that the holonomy representation of this group is  $\mathbb{Z}$ -equivalent to a direct product of representations of dimensions 1 and 3, and the 3-dimensional representation is the example  $\rho_6$  from [2, Section 5]. It was not known if there was a special free  $\mathbb{Z}[\mathbb{Z}^3]$ -resolution of  $\mathbb{Z}$  that admitted a compatible action of  $\mathbb{Z}_4$  via the representation  $\rho_6$  (see [2, 2.4, 5.1]). The example of the group  $1 : 4 \rtimes 4$  shows such a compatible action can never exist. Otherwise, by Lemma 2.2 and Theorem 2.3 in [2], we would arrive at a contradiction. In fact, we can say more:

**Theorem 4.1.** *Consider  $L = \mathbb{Z}^3$  and  $\Gamma = L \rtimes_{\rho_6} \mathbb{Z}_4$ . Let  $A = L$  as a  $\mathbb{Z}\Gamma$ -module via the representation  $\rho_6$ . Then, in the Lyndon-Hochschild-Serre spectral sequence associated to  $\Gamma$ , the differential  $d_2^{0,2}(A) : E_2^{0,2}(A) \rightarrow E_2^{2,1}(A)$  is nonzero. In particular,*

$$H^2(\Gamma, A) \not\cong \bigoplus_{i+j=2} H^i(\mathbb{Z}_4, H^j(L, A)).$$

The group  $\Gamma$  gives the lowest possible counterexample to a more general form of Conjecture 1.1 stated in [10, 1.2], where one allows nontrivial coefficients. This is because any group  $\mathbb{Z}^n \rtimes \mathbb{Z}_m$  for  $n \leq 2$  admits a local compatible action (see [1, 3.1]) and therefore, by a slight generalization of a theorem of Adem and Pan (see the proof of [1, 2.3]), satisfies this more general form of the conjecture.

*Proof of 4.1.* We apply the theory of characteristic classes introduced by Sah in [11] and further studied in [10] and [5].

Suppose, by a way of contradiction that  $d_2^{0,2}(A) = 0$ . One can easily check that  $H_2(A, \mathbb{Z})$  and  $H_3(A, \mathbb{Z})$ , as  $\mathbb{Z}\Gamma$ -modules, are isomorphic to  $A$  and the trivial module  $\mathbb{Z}$ , respectively.

Now, the characteristic class  $v_2^2$ , being in the image of the differential  $d_2^{0,2}(H_2(A, \mathbb{Z}))$ , vanishes. The only other possible nonzero characteristic class that can occur on the second page of the spectral sequence is  $v_2^3$ . But, by Theorem 7.11 of [5], it follows that the order of  $v_2^3$  is a divisor of one. Hence, it also vanishes. Since, we already know that the Lyndon-Hochschild-Serre spectral sequence associated to  $\Gamma$  collapses with  $\mathbb{Z}$ -coefficients, we can conclude that the differential  $d_3^{0,3}(H_3(A, \mathbb{Z})) = 0$  implying that  $v_3^3 = 0$ . Thus, we have shown that all characteristic classes vanish. Therefore, the Lyndon-Hochschild-Serre spectral sequence collapses at  $E_2$  for all coefficient modules that have a trivial  $L$ -action (see [5, 7.13]).

Since the holonomy representation of the group  $1 : 4 \rtimes 4$  decomposes into a direct sum of  $\rho_6$  and the nontrivial one-dimensional representation, by Corollary 4.2 of [10] (see also [5, 7.3-5]), it follows that the Lyndon-Hochschild-Serre spectral sequence associated to the group  $1 : 4 \rtimes 4$  collapses at  $E_2$  for all coefficient modules that have a trivial  $L$ -action. But this is clearly a contradiction to our computations of the 4-dimensional integral cohomology of the group  $1 : 4 \rtimes 4$  (see Remark 3).  $\square$

The second 4-dimensional counterexample to the conjecture is the crystallographic group of Section 3 given by the matrix (3.1). We enclose its cohomology groups in Table 1 under the number 2.

In dimension 5, there are 95 non-isomorphic crystallographic groups with cyclic holonomy of split type. Out of these, 6 do not satisfy Conjecture 1.1, 5 of them with holonomy  $\mathbb{Z}_4$  and 1 with holonomy  $\mathbb{Z}_8$ . We list the matrices corresponding to the their holonomy generators below and their cohomology groups in Table 1 with numbers from 3 to 8.

$$\begin{aligned}
3: & \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, & 4: & \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\
5: & \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}, & 6: & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \\
7: & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}, & 8: & \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}
\end{aligned}$$

TABLE 1. Cohomology groups of counterexamples up to dimension 5.

	Type	CARAT name	$H^1$	$H^2$	$H^3$	$H^4$	$H^5$	$H^{2k}$	$H^{2k+1}$
1	$4 \times 4$	min.27-1.2	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2^3$
2	$4 \times 4$	min.27-1.5	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_4^2$	$\mathbb{Z}_2^4$
3	$5 \times 4$	min.81-1.2	$\mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_2^5$	$\mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^6$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2^6$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2^6$
4	$5 \times 4$	min.81-1.5	$\mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^2$	$\mathbb{Z} \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_2^3$	$\mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^4$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2^4$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2^4$
5	$5 \times 4$	min.82-1.3	$\mathbb{Z}^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_2^2$	$\mathbb{Z} \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_2^4$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2^6$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2^6$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2^6$
6	$5 \times 4$	min.82-1.5	$\mathbb{Z}^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_2^2$	$\mathbb{Z} \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2^4$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2^4$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2^4$
7	$5 \times 4$	min.82-1.7	$\mathbb{Z}^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4^2$	$\mathbb{Z} \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2^5$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2^5$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2^5$
8	$5 \times 8$	min.142-1.2	0	$\mathbb{Z}^2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\mathbb{Z}^2$	$\mathbb{Z} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8^2 \oplus \mathbb{Z}_4^2$	$\mathbb{Z}_2^2$

Notation  $d \times n$  in the column “Type” gives the information that the group is of dimension  $d$  and has holonomy group of order  $n$ . CARAT name is the name of the group in the classification given in system CARAT. Note that CARAT uses left action of the holonomy group, thus holonomy representation has to be transposed before identification in CARAT.

*Remark 5.* Let  $\Gamma = L \rtimes G$  be the group  $8 : 5 \times 8$  from the table. We calculate the terms comprising the right hands side of the conjectured isomorphism:

$$H^i(G, H^j(L, \mathbb{Z})) = \begin{cases} \mathbb{Z} & j = 0, 4, i = 0 \\ \mathbb{Z}^2 & j = 2, 3, i = 0 \\ \mathbb{Z}_8 & j = 0, i \geq 1, 2|i \\ \mathbb{Z}_4 & j = 1, 2 \nmid i \\ \mathbb{Z}_2^2 & j = 2, 3, i \geq 1, 2|i \\ \mathbb{Z}_4 & j = 4, i \geq 1, 2|i \\ \mathbb{Z}_2 & j = 5, i \geq 1, 2|i \\ 0 & \text{otherwise} \end{cases}$$

to observe that the free ranks and the orders of the maximal finite subgroups of the groups  $H^k(L \rtimes G, \mathbb{Z})$  and  $\bigoplus_{i+j=k} H^i(G, H^j(L, \mathbb{Z}))$  are the same for every  $k$ .

This means that the Lyndon-Hochschild-Serre spectral sequence collapses at  $E_2$  but there are extension problems.

## 5. OTHER EXAMPLES

In this section we state the results of our computations for several examples of crystallographic groups of higher dimensions.

**5.1. Unresolved cases from [2].** Several crystallographic groups that were considered in [2] were not known to satisfy Conjecture 1.1. In section 5 of the same paper, the authors studied all crystallographic groups with holonomy  $\mathbb{Z}_4$  of split type whose holonomy representations are indecomposable. Out of total 9 such groups, there were two 4 dimensional examples, encoded  $\rho_8$  and  $\rho_9$ , which were not known to satisfy Conjecture 1.1. We verify that the example of  $\rho_8$  satisfies the conjecture. The example of  $\rho_9$  is the same as the one considered in Section 3, so also  $2 : 4 \times 4$  in Table 1. Hence, it does not satisfy the conjecture.

In section 6 of [2], in relation to certain 6-dimensional Calabi-Yau toroidal orbifolds arising in string theory, some crystallographic groups were considered. It was shown, that out of possible 18 such groups only two, denoted  $\mathbb{Z}_8^{(5)}$  and  $\mathbb{Z}_{12}^{(6)}$  were not known to admit local compatible actions. So, their cohomology was not computed. The 5-dimensional group  $\mathbb{Z}_8^{(5)}$  is the same as example  $8 : 5 \times 8$  from Table 1. So, it does not satisfy Conjecture 1.1. We show that the 6-dimensional group  $\mathbb{Z}_{12}^{(6)}$  also does not satisfy the conjecture. It has holonomy group  $G$  of order 12 generated by the matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Cohomology groups of the corresponding crystallographic group are as follows:

$$H^i(\mathbb{Z}^6 \rtimes G) = \begin{cases} 0 & i = 1 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_3 & i = 2 \\ \mathbb{Z}^2 & i = 3 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_3^3 & i = 4 \\ \mathbb{Z}_2^2 & i = 5 \\ \mathbb{Z} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6^2 \oplus \mathbb{Z}_3^5 & i = 6 \\ \mathbb{Z}_2^2 & i = 2k - 1, \quad k \geq 4 \\ \mathbb{Z}_{12}^2 \oplus \mathbb{Z}_6^2 \oplus \mathbb{Z}_3^5 & i = 2k, \quad k \geq 4. \end{cases}$$

**5.2. Cyclic holonomy group of odd non-prime order.** All previous counterexamples to Conjecture 1.1 have holonomy of order divisible by 4. We provide counterexample with odd order holonomy  $\mathbb{Z}_9$ .

The first occurrence of a crystallographic group with holonomy  $\mathbb{Z}_9$  is in dimension 6 and up to an isomorphism, it is the unique one in this dimension. We verify that this example satisfies the conjecture.

We find a counterexample of dimension 8 where the holonomy representation is generated by the matrix:

$$\begin{bmatrix} -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

by calculating that

$$H^4(\mathbb{Z}^8 \rtimes G, \mathbb{Z}) = \mathbb{Z}^8 \oplus \mathbb{Z}_9^2 \oplus \mathbb{Z}_3^4 \neq \mathbb{Z}^8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3^6 = \bigoplus_{i+j=4} H^i(G, H^j(\mathbb{Z}^8, \mathbb{Z})).$$

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